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LETTER TO THE EDITOR

Dobiński-type relations and the log-normal distribution

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Abstract
We consider sequences of generalized Bell numbers $B(n), n = 1, 2, \ldots,$ which can be represented by Dobiński-type summation formulae, i.e. $B(n) = \frac{1}{C}\sum_{k=0}^{\infty}\frac{[P(k)]^n}{D(k)}$, with $P(k)$ a polynomial, $D(k)$ a function of k and $C = \text{const}$. They include the standard Bell numbers $(P(k) = k, D(k) = k!, C = e)$, their generalizations $B_{r,r}(n), r = 2, 3, \ldots,$ appearing in the normal ordering of powers of boson monomials $(P(k) = \frac{(k+r)!}{k!}, D(k) = k!, C = e)$, variants of 'ordered' Bell numbers $B_o^{(p)}(n)$ $(P(k) = k, D(k) = \left(\frac{p+1}{p}\right)^k, C = 1 + p, p = 1, 2 \ldots)$, etc. We demonstrate that for α, β, γ, t positive integers $(\alpha, t \neq 0)$, $[B(\alpha n^2 + \beta n + \gamma)]^t$ is the nth moment of a positive function on $(0, \infty)$ which is a weighted infinite sum of log-normal distributions.

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In a recent investigation [1] we analysed sequences of integers which appear in the process of normal ordering of powers of monomials of boson creation $a^\dagger$ and annihilation a operators, satisfying the commutation rule $[a, a^\dagger] = 1$. For r, s integers such that $r \geqslant s$, we define the generalized Stirling numbers of the second kind $S_{r,s}(n, k)$ as

$$[(a^\dagger)^r a^s]^n = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n, k)(a^\dagger)^k a^k \tag{1}$$

and the corresponding Bell numbers $B_{r,s}(n)$ as

$$B_{r,s}(n) = \sum_{k=s}^{ns} S_{r,s}(n, k). \tag{2}$$

In [1] explicit and exact expressions for $S_{r,s}(n, k)$ and $B_{r,s}(n)$ were found. In a parallel study [2] it was demonstrated that $B_{r,s}(n)$ can be considered as the nth moment of a probability distribution on the positive half-axis. In addition, for every pair (r, s) the corresponding distribution can be explicitly written down. These distributions constitute the solutions of a family of Stieltjes moment problems, with $B_{r,s}(n)$ as moments. Of particular interest to us are the sequences with $r = s$, for which the following representation as an infinite series has been obtained:

$$B_{r,r}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{(k+r)!}{k!} \right]^{n-1} \tag{3}$$

$$= \frac{1}{e} \sum_{k=0}^{\infty} \frac{[k(k+1)\ldots(k+r-1)]^n}{(k+r-1)!} \qquad n > 0. \tag{4}$$

Equations (3) and (4) are generalizations of the celebrated Dobiński formula ($r = 1$) [3]:

$$B_{1,1}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} \qquad n > 0 \tag{5}$$

which expresses the conventional Bell numbers $B_{1,1}(n)$ as a rapidly convergent series. Its simplicity has inspired combinatorialists such as Rota [4] and Wilf [5]. Equation (5) has far-reaching implications in the theory of stochastic processes [6–8].

The probability distribution whose nth moment is $B_{r,r}(n)$ is an infinite ensemble of weighted Dirac delta functions located at a specific set of integers (a so-called *Dirac comb*):

$$B_{r,r}(n) = \int_0^{\infty} x^n \left\{ \frac{1}{e} \sum_{k=0}^{\infty} \frac{\delta(x - k(k+1)\ldots(k+r-1))}{(k+r-1)!} \right\} \mathrm{d}x \qquad n > 0. \tag{6}$$

For $r = 1$, the discrete distribution of equation (6) is the weight function for the orthogonality relation for Charlier polynomials [9]. In contrast we emphasize that for $r \neq s$ the $B_{r,s}(n)$ are moments of continuous distributions [2].

In this letter we wish to point out an intimate relation between the formulae of equations (3)–(5) and the log-normal distribution [10, 11]:

$$P_{\sigma,\mu}(x) = \frac{1}{\sqrt{2\pi}\sigma x} \mathrm{e}^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}} \qquad x \geqslant 0 \quad \sigma, \mu > 0. \tag{7}$$

First we quote the standard expression for its nth moment:

$$M_n = \int_0^{\infty} x^n P_{\sigma,\mu}(x)\,\mathrm{d}x = \mathrm{e}^{n(\mu + n\frac{\sigma^2}{2})} \qquad n \geqslant 0 \tag{8}$$

which can be reparametrized for $k > 1$ as

$$M_n = k^{\alpha n^2 + \beta n} \tag{9}$$

with

$$\mu = \beta \ln(k) \tag{10}$$

$$\sigma = \sqrt{2\alpha \ln(k)} > 0. \tag{11}$$

Given three integers α, β, γ (where $\alpha > 0$), we wish to find a weight function $W_{1,1}(\alpha, \beta, \gamma; x) > 0$ such that

$$B_{1,1}(\alpha n^2 + \beta n + \gamma) = \int_0^{\infty} x^n W_{1,1}(\alpha, \beta, \gamma; x)\,\mathrm{d}x. \tag{12}$$

Equations (5), (7) and (9) provide an immediate solution:

$$W_{1,1}(\alpha,\beta,\gamma;x) = \frac{1}{e}\left[\delta(x-1) + \sum_{k=2}^{\infty} \frac{k^{\gamma}\exp\left(-\frac{(\ln(x)-\beta\ln(k))^2}{4\alpha\ln(k)}\right)}{2xk!\sqrt{\pi\alpha\ln(k)}}\right] \tag{13}$$

which is an *infinite* sum of weighted log-normal distributions supplemented by a single Dirac peak of weight e^{-1} located at $x = 1$. Thus it is a *superposition* of discrete and continuous distributions. Virtually the same approach can be adopted for the sequences $B_{r,r}(n)$, $r > 1$. In this case the $k = 1$ term in the numerator of equation (3) is larger than one and so there will be no Dirac peak in the formula. Then the function $W_{r,r}(\alpha,\beta,\gamma;x) > 0$ defined by (α,β,γ integers, $\alpha,\gamma > 0$)

$$B_{r,r}(\alpha n^2+\beta n+\gamma) = \int_0^{\infty} x^n W_{r,r}(\alpha,\beta,\gamma;x)\,\mathrm{d}x \tag{14}$$

is a purely *continuous* probability distribution given again by an infinite sum of weighted log-normal distributions:

$$W_{r,r}(\alpha,\beta,\gamma;x) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{\left[\frac{(k+r)!}{k!}\right]^{\gamma-1}\exp\left(-\frac{\left[\ln(x)-\beta\ln\left(\frac{(k+r)!}{k!}\right)\right]^2}{4\alpha\ln\left[\frac{(k+r)!}{k!}\right]}\right)}{2xk!\sqrt{\pi\alpha\ln\left[\frac{(k+r)!}{k!}\right]}}. \tag{15}$$

The solutions of the moment problems of equations (9), (12) and (14) are not unique. More general solutions may be obtained by the method of the inverse Mellin transform, see [12].

Several other types of combinatorial sequences have properties exemplified by equations (12) and (14). We quote, for example, the so-called 'ordered' Bell numbers $B_o(n)$ defined as [5]

$$B_o(n) = \sum_{k=1}^{n} S(n,k)k! \tag{16}$$

where $S(n,k)$ are the Stirling numbers of the second kind, $S_{1,1}(n,k)$ in our notation. These ordered Bell numbers satisfy the following Dobiński-type relation:

$$B_o(n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k} \tag{17}$$

from which a formula analogous to equation (14) readily follows. A more general identity of type (17) is [13]

$$B_o^{(p)}(n) = \frac{1}{p+1}\sum_{k=1}^{\infty} k^n\left(\frac{p}{p+1}\right)^k = \sum_{k=0}^{n} S(n,k)k!p^k \qquad p = 2,3,\ldots. \tag{18}$$

We will not discuss other types of sequences but rather observe that the relations of equations (3), (4), (5), (17), (18) naturally imply that any power of these numbers also satisfies a Dobiński-type relation. As an example we give explicitly the simplest case of equation (5). For integer $t > 0$:

$$[B_{1,1}(n)]^t = \frac{1}{e^t}\sum_{k_1,k_2,\ldots,k_t=0}^{\infty}\frac{(k_1k_2\ldots k_t)^n}{k_1!k_2!\ldots k_t!} \tag{19}$$

with correspondingly more complicated formulae of a similar nature for powers of $B_{r,r}(n)$, $B_o(n)$ and $B_o^{(p)}(n)$. For combinatorial applications of equation (19), see [14–16].

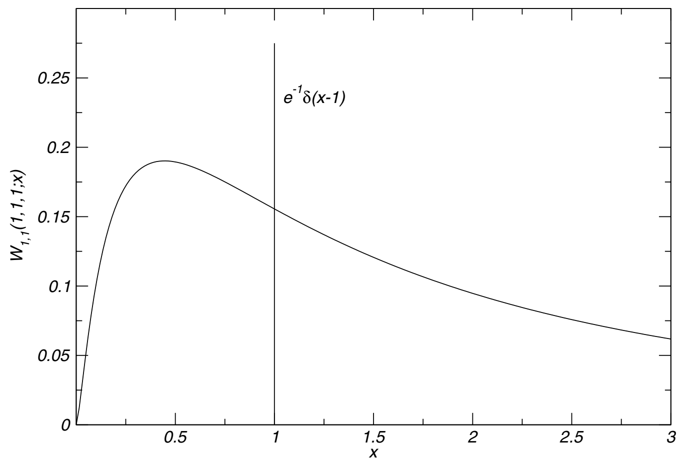

Figure 1. Weight function $W_{1,1}(1, 1, 1; x)$, see equation (13).

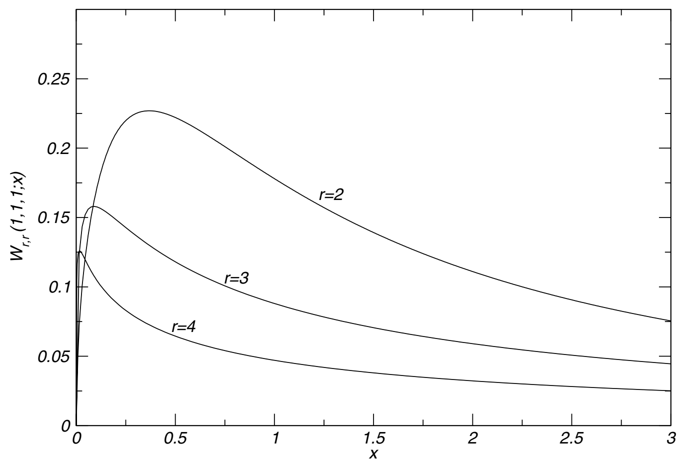

Figure 2. Weight functions $W_{r,r}(1, 1, 1; x)$ for $r = 2, 3, 4$.

We conclude that for any sequences of the type $B(n)$ specified above $[B(\alpha n^2 + \beta n + \gamma)]^t$ is always given as an nth moment of a positive function on $(0, \infty)$ expressible by sums of weighted log-normal distributions. We illustrate such a function for $B_{1,1}(n)$ in figure 1. The application to $B_{r,r}(n)$ for $r = 2, 3, 4$ is presented in figure 2. The area under every curve is equal to 1 on extrapolating to large x (not displayed). In both examples we have chosen $\alpha = \beta = \gamma = 1$. Observe the exceedingly slow decrease of these probabilities for $x \to \infty$. This is confirmed by the fact that the moment sequences $[B(\alpha n^2 + \beta n + \gamma)]^t$ are extremely rapidly increasing. In the simplest case $\alpha = \beta = \gamma = t = 1$ we find $B_{1,1}(n^2 + n + 1) = 1, 5, 877, 27\,644\,437, 474\,869\,816\,156\,751$ for $n = 0, \ldots, 4$.

The circumstance that we can determine the positive solutions of the Stieltjes moment problem with both $B(n)$ (discrete distribution) and $[B(\alpha n^2+\beta n+\gamma)]^t$ (continuous distribution) is a very specific consequence of the existence of Dobiński-type expansions. To our knowledge it has no equivalent in standard solutions of the moment problem. For instance, if the moments are $n!$ the solution e^{-x} does not give any indication as to how one might obtain the solution for the moments equal to $(n^2)!$.

The strict positivity of $W_{r,r}(\alpha, \beta, \gamma; x)$, for $r = 1, 2, \ldots,$ suggests their use in the construction of coherent states, which satisfy the *resolution of identity* property [17–20]. This can be done by the substitution $n! \to B_{r,r}(\alpha n^2 + \beta n + \gamma)$ in the definition of standard coherent states. More precisely, for a complete and orthonormal set of wavefunctions $|n\rangle$ such that $\langle n|n'\rangle = \delta_{n,n'}$ and complex z we define the normalized coherent state as

$$|z; \alpha, \beta, \gamma\rangle = \frac{1}{\mathcal{N}^{1/2}(\alpha, \beta, \gamma; |z|^2)} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{B_{r,r}(\alpha n^2 + \beta n + \gamma)}} |n\rangle \tag{20}$$

with the normalization

$$\mathcal{N}(\alpha, \beta, \gamma; x) = \sum_{n=0}^{\infty} \frac{x^n}{B_{r,r}(\alpha n^2 + \beta n + \gamma)} \tag{21}$$

which is a rapidly converging function of x for $0 \leqslant x < \infty, x = |z|^2$. Then, using the procedure of [18] we can demonstrate that the states of equation (20) along with equation (14) automatically satisfy the resolution of unity

$$\int\!\!\int_{\mathbb{C}} d^2z |z; \alpha, \beta, \gamma\rangle \tilde{W}_{r,r}(\alpha, \beta, \gamma; |z|^2) \langle z; \alpha, \beta, \gamma| = I = \sum_{n=0}^{\infty} |n\rangle\langle n| \tag{22}$$

with

$$W_{r,r}(\alpha, \beta, \gamma; |z|^2) = \pi \frac{\tilde{W}_{r,r}(\alpha, \beta, \gamma; |z|^2)}{\mathcal{N}(\alpha, \beta, \gamma; |z|^2)}. \tag{23}$$

We are currently investigating the quantum-optical properties of states defined in equation (20).

We close by quoting from [7] that, ‘the idea of representing the combinatorially defined numbers by an infinite sum or an integral, typically with a probabilistic interpretation, has proved to be a very fruitful one’. In our particular case it has allowed us to reveal quite an unexpected relation between the Dobiński-type summation relations, which by themselves are reflections of boson statistics, and the log-normal distribution.

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